Note on the position of ring singularities in an axisymmetric potential field

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SUMMARY

In this work a method is given for finding the position of ring singularities in a three-dimensional potential field having axial symmetry, by consideration of the very much easier case of a similar two-dimensional potential function. It is seen that the traces of these ring singularities on a plane through the axis of symmetry occur at points corresponding to those of the singularities existing in the two-dimensional plane when the axial velocity potential functions are the same. It is thought that this might be of value in the plotting of stream surfaces.

INTRODUCTION

When an arbitrary velocity potential function for perfect fluid flow is assumed, it is essential that the singularities of this function or its derivatives should be excluded from the fluid region that it is hoped may be reproduced physically. In the following work a method has been developed for obtaining the positions of the singularities in velocities, or velocity potential, by consideration of similar velocity or velocity potential distributions in the two-dimensional case. It will be supposed that the velocity potential function, or one of its derivatives, is known along the axis of symmetry, and it can be shown that this defines the velocity potential function throughout all space.

First it will be shown that a relationship exists between the two-dimensional case and the three-dimensional case with axial symmetry.

The procedure for then obtaining the ring singularities when $\phi = f(x)$ along the axis in three dimensions, with symmetry assumed, will be to consider the position of the singularities existing when $\phi = f(x)$ along the x-axis in two dimensions.

SIMILARITY OF CERTAIN TWO-DIMENSIONAL AND THREE-DIMENSIONAL VELOCITY POTENTIAL FUNCTIONS

The axis of symmetry will be taken as the x-axis in both the two- and three-dimensional cases. ϕ denotes a velocity potential function in both cases, and the value of ϕ along the axis of symmetry is supposed to be $2f(x)$.

If y is defined as the coordinate measured perpendicular to the x-axis in two dimensions and $z = x+iy$, it follows that $f(z) = \bar{f}(z)$, and so

$$\phi = f(z)+f(\bar{z}) \tag{1}$$

at any point of the two-dimensional field, with the usual notation for conjugate complex quantities. Here, $f(z)$ is the analytic function of z that takes the real value $f(x)$ when $y = 0$.

For the three-dimensional case with axial symmetry, the definition of the axial velocity potential as $2f(x)$ determines the potential function throughout the whole of space except along a number of lines. It may be shown (Whittaker & Watson 1927, p. 388) that

$$\phi = \frac{2}{\pi}\int_0^{\pi} f(x+ir\cos\theta)\,d\theta \tag{2}$$

at every point of the field at which the integral is analytic, where r is defined as the coordinate measured perpendicular to the x-axis in three dimensions. Putting $\lambda = \cos\theta$ in (2), one obtains

$$\phi = \frac{2}{\pi}\int_{-1}^{1}\frac{f(x+ir\lambda)}{\sqrt{(1-\lambda^2)}}\,d\lambda = \frac{2}{\pi}\left[\int_0^1 \frac{f(x+ir\lambda)}{\sqrt{(1-\lambda^2)}}\,d\lambda + \int_0^1 \frac{f(x-ir\lambda)}{\sqrt{(1-\lambda^2)}}\,d\lambda\right]. \tag{3}$$

There is clearly great similarity between equations (1) and (3).

Suppose that $f(z)$ has a singularity at $z = \alpha+i\beta$; then $\lambda = 1$ gives the smallest value of r for which $\lambda r = \beta$, and when $\lambda = 1$, $f(x+ir\lambda)$ has a singularity at $x = \alpha$, $r = \beta$.

When no finite limit $f(\alpha+i\beta)$ exists, the integral (3) may not converge and then ϕ will be infinite at the singularity $x = \alpha$, $r = \beta$; note that the traces of this singularity are at the same points on a plane through the axis of symmetry as in the two-dimensional case. If however the integral (3) converges for the range $\lambda^2 \leqslant 1$ the velocity potential still exists. The condition that the integral (3) converges at $\lambda = 1$ is that

$$\lim_{\lambda\to 1}\frac{(1-\lambda)f(\alpha+i\beta)}{\sqrt{(1-\lambda^2)}}$$

is finite. Even in this case, however, it is easily seen that higher derivatives of ϕ must become infinite, so that the point is still a singularity.

Taking the velocity in the direction of the x-axis to be u $(= -\partial\phi/\partial x)$, the preceding work may be repeated with $f(x)$ replaced by $-f'(x)$. Hence the singularities in the velocity distributions in the field follow a similar pattern.

Some examples given below illustrate this.

Example 1

Suppose the axial velocity is given as $(1+x^2)^{-n} = -2f'(x)$, say.

Then, in two dimensions,

$$u = -\frac{\partial\phi}{\partial x} = \tfrac{1}{2}[(1+z^2)^{-n}+(1+\bar{z}^2)^{-n}].$$

This becomes infinite for $n > 0$ when $z^2+1 = 0$, or $y = \pm 1$, $x = 0$.

In three dimensions

$$f'(\alpha+i\beta\lambda) = -\tfrac{1}{2}[1+\alpha+i\beta\lambda)^2]^{-n}.$$

In this case

$$\lim_{\lambda\to 1}\frac{-(1-\lambda)^{1/2}}{2[1+(\alpha+i\beta\lambda)^2]^n} = \infty$$

for $n-\frac{1}{2} > 0$ when $\alpha = 0$, $\beta = 1$.

Hence a ring singularity exists symmetrically about the x-axis, of radius $r = 1$ in the plane $x = 0$, and the value of u is not finite at points on this ring provided $n - \frac{1}{2} > 0$. For the special case when $n = 1$, the integral (3) becomes

$$\frac{2}{\pi}\left[\int_0^1 \frac{d\lambda}{[(x+ir\lambda)^2+1]\sqrt{(1-\lambda^2)}} + \int_0^1 \frac{d\lambda}{[(x-ir\lambda)^2+1]\sqrt{(1-\lambda^2)}}\right],$$

and this can be found explicitly as

$$u(x,r) = \frac{1}{2i}\left[\{r^2+(v-i)^2\}^{-1/2} - \{r^2+(x+i)^2\}^{-1/2}\right],$$

which clearly has a singularity at $x = 0$, $r^2 = 1$. It is, in addition, discontinuous across the whole 'branch line' $x = 0$, $r > 1$.

Example 2

Suppose the axial velocity is given as $(1+e^x)^{-n} = -2f'(x)$, say.

Then, in two dimensions, $u = \frac{1}{2}\{(1+e^z)^{-n} + (1+e^{\bar{z}})^{-n}\}$; this becomes infinite for $n > 0$ when $e^z + 1 = 0$, or $x = 0$, $y = (2K+1)\pi$, where K is an integer.

For three dimensions, $f'(\alpha + i\beta\lambda) = -\frac{1}{2}(1+e^{\alpha+i\beta\lambda})^{-n}$ and

$$\lim_{\lambda\to 1} \frac{-1(-\lambda)^{1/2}}{2[1+e^{\alpha+i\beta\lambda}]^n} = \infty$$

for $n - \frac{1}{2} > 0$ when $\alpha = 0$ and $\beta = (2K+1)\pi$.

Hence ring singularities exist symmetrically about the x-axis, of radii $r = \pi, 3\pi, 5\pi, \ldots (2K+1)\pi$, in the plane $x = 0$.

A variation of this axial velocity has been used by Szczeniowski (1943) for designing wind tunnel contractions; but he does not appear to know where these singularities occur.

Example 3

Suppose the axial velocity is given as $2(1+x^4)^{-1} = -2f'(x)$, say.

In two dimensions, $u = (1+z^4)^{-1} + (1+\bar{z}^4)^{-1}$, which ceases to be analytic when $z^4 + 1 = 0$, that is, when $z = \exp\{\frac{1}{4}i(2K+1)\pi\}$, where K is an integer and $x + iy = (\pm 1 \pm i)/\sqrt{2}$.

For the three-dimensional case,

$$f'(\alpha + i\beta\lambda) = -\{1+(\alpha+i\beta\lambda)^4\}^{-1},$$

and for $\alpha = \pm 1/\sqrt{2}$, $\beta^2 = \frac{1}{2}$

$$\lim_{\lambda\to 1}\left\{\frac{-(1-\lambda)^{1/2}}{1+(\alpha+i\beta\lambda)^4}\right\} = \lim_{\lambda\to 1}\left\{\frac{-4(1-\lambda)^{1/2}}{4+(1\pm i\lambda)^4}\right\} = \infty.$$

Hence for this axial velocity there exist two ring singularities, given by $x = \pm 1/\sqrt{2}$, $r^2 = \frac{1}{2}$, at which the value of u is not bounded.

Conclusions

The procedure of defining an axial velocity distribution in three dimensions has been used by several authors in the design of nozzles and wind tunnel contractions. Prior to this work, it has been necessary to calculate the stream functions (or potential functions) and then deduce the regions of the fluid in which the velocity components tend to infinity. Since these functions have often been in series form this has not always been easy and has usually been tedious. It is suggested here that, following the adoption of the axial velocity potential function, the position of the singularities in the two-dimensional (and hence of the three-dimensional) field should be found, as this will give a good indication of the regions where sharp rises in velocity along stream surfaces may occur. If, then, it is of importance to eliminate such rises, the plotting of a few stream surfaces in the region of the singularities alone might prevent a considerable wastage of time.

A point of interest is the reduction in degree of the singularities occurring in the three-dimensional case. For example, when the axial velocity is $(1+x^2)^{-2}$, then on the plane $x = 0$ the value of the $u(0,y)$ in two dimensions is $(1-y^2)^{-2}$ whilst it can be shown that the value of $u(0,r)$ in three dimensions is

$$\tfrac{1}{2}\{(1-r^2)^{-3/2}+(1-r^2)^{-1/2}\}.$$

The 'order of infinity' has therefore dropped from 2 to $\frac{3}{2}$; and a reduction in order of $\frac{1}{2}$ (compare example 1) seems general when going from three to two dimensions.

Whilst the preceding work was done mainly for the consideration of perfect fluid flow, it is quite general and could be applied to other problems with axial symmetry.

References

Szczeniowski, B. 1943 On the contraction cone for a wind tunnel, *J. Aero. Sci.* **10.**

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